

Different Asymptotic Spreading Speeds Induced by Advection in a Diffusion Problem with Free Boundaries *

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Abstract

In this paper, we consider a Fisher-KPP equation with an advection term and two free boundaries, which models the behavior of an invasive species in one dimension space. When spreading happens (that is, the solution converges to a positive constant), we use phase plane analysis and upper/lower solutions to prove that the rightward and leftward asymptotic spreading speeds exist, both are positive constants. Moreover, one of them is bigger and the other is smaller than the spreading speed in the corresponding problem without advection term.

1 Introduction

In 2010, Du and Lin [6] studied the following Fisher-KPP problem with free boundaries:

$$\begin{cases} u_t - du_{xx} = u(1 - u), & g(t) < x < h(t), \ t > 0, \\ u(t, g(t)) = 0, \ g'(t) = -\mu u_x(t, g(t)), & t > 0, \\ u(t, h(t)) = 0, \ h'(t) = -\mu u_x(t, h(t)), & t > 0, \\ -g(0) = h(0) = h_0, \ u(0, x) = u_0(x), & -h_0 \leq x \leq h_0, \end{cases} \quad (P_0)$$

where d and μ are positive constants, the initial function $u_0(x)$ satisfies

$$u_0 \in C^2([-h_0, h_0]), \ u_0(\pm h_0) = 0 \text{ and } u_0 > 0 \text{ in } (-h_0, h_0), \quad (1.1)$$

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for some $h_0 > 0$. They used (P_0) to model the spreading of a new or invasive species with population density $u(t, x)$ over a one dimensional habitat, with the free boundaries $x = g(t)$, $h(t)$ representing the expanding fronts. They obtained a dichotomy result, that is, either spreading happens ($u(t, \cdot) \rightarrow 1$ locally uniformly in \mathbb{R} and $h(t), -g(t) \rightarrow \infty$ as $t \rightarrow \infty$) or vanishing happens ($u(t, \cdot) \rightarrow 0$ uniformly in $[g(t), h(t)]$ as $t \rightarrow \infty$ and $h(t) - g(t) < \infty$). Furthermore, when spreading happens, they obtained the existence of the asymptotic spreading speed ([6, Proposition 4.1]):

$$c^* := \lim_{t \rightarrow \infty} \frac{h(t)}{t} = \lim_{t \rightarrow \infty} \frac{-g(t)}{t} > 0. \quad (1.2)$$

Recently, further extensions have been done, for example, Du and Guo [4, 5] studied the problem in higher dimension spaces and in heterogeneous environment. Du and Lou [7] studied the problem with general nonlinear f , including general monostable, bistable and combustion types of f . Among others, they all proved that the asymptotic spreading speed when spreading happens is the same positive constant in any direction.

However, some species prefers to move towards one direction because of rich resource, appropriate climate, etc. Some diseases spread along the wind direction. In 2009, Maidana and Yang in [9] studied the propagation of West Nile Virus from New York City to California state. It was observed that West Nile Virus appeared for the first time in New York City in the summer of 1999. In the second year the wave front travels 187km to the north and 1100km to the south. Therefore, they took account of the advection movement and showed that bird advection becomes an important factor for lower mosquito biting rates. Recently, Averill in [1] considered the effect of intermediate advection on the dynamics of two-species competition system, and provides a concrete range of advection strength for the coexistence of two competing species. Moreover, three different kinds of transitions from small advection to large advection were illustrates theoretically and numerically.

What is the difference between the asymptotic spreading speed of the left frontier and that of the right frontier when invasive species is spreading? To

address the question, in this paper we study the following problem with an advection term:

$$(P_1) \quad \begin{cases} u_t - du_{xx} + \beta u_x = u(1 - u), & g(t) < x < h(t), \ t > 0, \\ u(t, g(t)) = 0, \ g'(t) = -\mu u_x(t, g(t)), & t > 0, \\ u(t, h(t)) = 0, \ h'(t) = -\mu u_x(t, h(t)), & t > 0, \\ -g(0) = h(0) = h_0, \ u(0, x) = u_0(x), & -h_0 \leq x \leq h_0, \end{cases}$$

where d, μ, h_0, u_0 are as above and $\beta > 0$ is a constant.

By a similar argument as in [4, 6, 7], we have the following basic results.

- (i) Problem (P_1) has a time global solution (u, g, h) with $u \in C^{1+\alpha/2, 2+\alpha}((0, \infty) \times [g(t), h(t)])$ and $g, h \in C^{1+\alpha/2}([0, \infty))$ for any $\alpha \in (0, 1)$;
- (ii) $0 < u(t, x) \leq C_1$ for $g(t) < x < h(t)$, $t > 0$ and $0 < -g'(t), h'(t) < C_2$ for $t > 0$, where C_1 and C_2 are constants independent of t .

In a forthcoming paper [8], we studied the asymptotic behavior of the solutions of (P_1) . More precisely, we gave some sufficient conditions for spreading and some sufficient conditions for vanishing. It turns out that spreading happens only if

$$0 < \beta < 2\sqrt{d}. \quad (1.3)$$

This paper is devoted to the difference between the leftward and rightward asymptotic spreading speeds induced by the advection term βu_x . This is an interesting problem from ecological point of view.

Theorem 1.1 *Assume $0 < \beta < 2\sqrt{d}$. Let (u, g, h) be a solution of (P_1) for which spreading happens. Then the leftward and rightward asymptotic spreading speeds exist:*

$$c_l^* := \lim_{t \rightarrow \infty} \frac{-g(t)}{t}, \quad c_r^* := \lim_{t \rightarrow \infty} \frac{h(t)}{t}.$$

Moreover, $0 < c_l^* < c^* < c_r^*$, where c^* is the spreading speed of the solution of (P_0) .

c^* is given in (1.2) which is nothing but k_0 in [6, Proposition 4.1], or c^* in [7, Theorem 1.10]. It depends on the parameter μ . Similarly c_l^* and c_r^* depend

on μ . Clearly, c_r^* and c_l^* also depend on β . On these dependence we have the following results.

Theorem 1.2 (i) *If $\beta \in (0, 2\sqrt{d})$ is fixed, then c_l^*, c^*, c_r^* are strictly increasing in μ , and*

$$\begin{aligned} \lim_{\mu \rightarrow 0} c_l^* &= 0, & \lim_{\mu \rightarrow \infty} c_l^* &= 2\sqrt{d} - \beta, \\ \lim_{\mu \rightarrow 0} c^* &= 0, & \lim_{\mu \rightarrow \infty} c^* &= 2\sqrt{d}, \\ \lim_{\mu \rightarrow 0} c_r^* &= 0, & \lim_{\mu \rightarrow \infty} c_r^* &= 2\sqrt{d} + \beta; \end{aligned}$$

(ii) *if μ is fixed, then $c_r^*, -c_l^*$ are strictly increasing in β , and*

$$\lim_{\beta \rightarrow 0} c_l^* = \lim_{\beta \rightarrow 0} c_r^* = c^*, \quad \lim_{\beta \rightarrow 2\sqrt{d}} c_l^* = 0.$$

2 Semi-waves and spreading speeds

Throughout this section we assume that (1.3) holds and that (u, g, h) is a solution of (P_1) for which spreading happens, that is, $h(t), -g(t) \rightarrow \infty$ ($t \rightarrow \infty$), and $u(t, \cdot) \rightarrow 1$ locally uniformly in \mathbb{R} . Denote $f(u) := u(1 - u)$ for convenience. We remark that the approaches below remain valid for general monostable non-linear f .

To determine the spreading speed, we will construct upper and lower solutions based on semi-waves.

2.1 Phase plane analysis and semi-waves

We call $q(z)$ a semi-wave with speed c if $(c, q(z))$ satisfies

$$\begin{cases} q'' - \frac{c-\beta}{d}q' + \frac{f(q)}{d} = 0 & \text{for } z \in (0, \infty), \\ q(0) = 0, \quad q(\infty) = 1, \quad q(z) > 0 & \text{for } z \in (0, \infty). \end{cases} \quad (2.1)$$

The first equation in this problem is equivalent to the following system:

$$\begin{cases} q' = p, \\ p' = \frac{c-\beta}{d}p - \frac{f(q)}{d}. \end{cases} \quad (2.2)$$

A solution $(q(z), p(z))$ of this system traces out a trajectory in the q, p -plane or, as it is usually called, the phase plane (cf. [2, 3, 7, 10]). Such a trajectory has

slope

$$\frac{dp}{dq} = \frac{c - \beta}{d} - \frac{f(q)}{dp} \quad (2.3)$$

at any point where $p \neq 0$. Here we are only interested in a trajectory of (2.2) that starts from the point $(0, \omega)$ with some $\omega \geq 0$ and ends at the point $(1, 0)$ as $z \rightarrow +\infty$.

For any fixed $c \geq 0$, $(0, 0)$ and $(1, 0)$ are critical points of the system (2.2). The eigenvalues of the corresponding linearizations are

$$\lambda_0^\pm = \frac{c - \beta \pm \sqrt{(c - \beta)^2 - 4d}}{2d} \quad (\text{at } (0, 0)),$$

$$\lambda_1^\pm = \frac{c - \beta \pm \sqrt{(c - \beta)^2 + 4d}}{2d} \quad (\text{at } (1, 0)),$$

respectively. Thus $(1, 0)$ is a saddle point and $(0, 0)$ is

- (i) a center or a spiral point, if $0 \leq c < \beta + 2\sqrt{d}$;
- (ii) a nodal point, if $c \geq \beta + 2\sqrt{d}$.

Therefore, by the theory of ODE (cf. [10]), there exactly two trajectories of (2.2) that approach $(1, 0)$ from $q < 1$. One of them, denoted by T_r^c , has slope $\lambda_1^- < 0$ at $(1, 0)$. Suppose that T_r^c is expressed by a function $p = P_r^c(q)$. Then $p = P_r^c(q)$ satisfies (2.3) and T_r^c lies in the semistrip

$$S = \{(q, p) : 0 < q < 1, p > 0\}.$$

T_r^c is a trajectory through $(1, 0)$ and $(0, P_r^c(0^+))$ for some $P_r^c(0^+) \geq 0$. The following are well known results (cf. [2, 3, 7, 11]).

Proposition 2.1 *Let $c_r^0 := 2\sqrt{d} + \beta$. Then*

- (i) *for any $c \in [0, c_r^0)$, $P_r^c(0)$ is positive, continuous, strictly decreasing in c , and $\lim_{c \nearrow c_r^0} P_r^c(0) = 0$;*
- (ii) *for any $c \geq c_r^0$, $P_r^c(0^+) = 0$.*

In case (ii), each T_r^c is a trajectory in S through $(0, 0)$ and $(1, 0)$, and so it corresponds to a traveling wave with speed c , c_r^0 is nothing but the minimal speed of these traveling waves.

Denote $\zeta(c) := P_r^c(0) - \frac{c}{\mu}$ for $c \in [0, c_r^0]$. In view of Proposition 2.1, $\zeta(c)$ is continuous and strictly decreasing in $c \in [0, c_r^0]$, and it satisfies

$$\begin{aligned}\zeta(0) &= P_r^0(0) > P_r^\beta(0) = \sqrt{\frac{2}{d} \int_0^1 f(s) ds} > 0, \\ \zeta((c_r^0)^-) &= -\frac{c_r^0}{\mu} < 0.\end{aligned}$$

Thus there exists a unique $c_r^* \in (0, c_r^0)$ such that $\zeta(c_r^*) = 0$, i.e. $P_r^{c_r^*}(0) = c_r^*/\mu$. Summarizing the above results we have the following proposition.

Proposition 2.2 *Problem (2.1) has exactly one solution $(c, q) = (c_r^*, q_r^*)$ such that*

$$\mu(q_r^*)'(0) = c_r^*. \quad (2.4)$$

Moreover, $c_r^* \in (0, \beta + 2\sqrt{d})$.

Later we will use this semi-wave to estimate the rightward spreading speed. Similarly, to estimate the leftward spreading speed, we will need another semi-wave traveling to left, which is a solution of the following problem:

$$\begin{cases} q'' - \frac{c+\beta}{d}q' + \frac{f(q)}{d} = 0 & \text{for } z \in (0, \infty), \\ q(0) = 0, \quad q(\infty) = 1, \quad q(z) > 0 & \text{for } z \in (0, \infty). \end{cases} \quad (2.5)$$

Similar as above, this problem can be studied by considering the problem

$$\frac{dp}{dq} = \frac{c+\beta}{d} - \frac{f(q)}{dp} \quad (2.6)$$

in the q, p -phase plane, where $p = q'$. Denote $P_l^c(q)$ the solution of this equation whose trajectory through $(1, 0)$ and $(0, P_l^c(0^+))$ for some $P_l^c(0^+) \geq 0$. In a similar way as above we have the following results.

Proposition 2.3 *Let $c_l^0 := 2\sqrt{d} - \beta$. Then*

- (i) *for any $c \in [0, c_l^0]$, $P_l^c(0)$ is positive, continuous, strictly decreasing in c , and $\lim_{c \nearrow c_l^0} P_l^c(0) = 0$;*

(ii) for any $c \geq c_l^0$, $P_l^c(0^+) = 0$.

Proposition 2.4 *Problem (2.5) has exactly one solution $(c, q) = (c_l^*, q_l^*)$ such that*

$$\mu(q_l^*)'(0) = c_l^*. \quad (2.7)$$

Moreover, $c_l^* \in (0, c_l^0)$.

Next, we make suitable perturbations of $f(u)$ to derive corresponding semi-waves that can be used to construct upper and lower solutions of (P_1) .

For any small $\varepsilon > 0$, set

$$\tilde{f}_\varepsilon(u) := f(u) - \frac{\varepsilon}{1-\varepsilon}u^2 \equiv u\left(1 - \frac{1}{1-\varepsilon}u\right),$$

$$\hat{f}_\varepsilon(u) := f(u) + \frac{\varepsilon}{1+\varepsilon}u^2 \equiv u\left(1 + \frac{1}{1+\varepsilon}u\right).$$

Note that $\tilde{f}_\varepsilon(u)$ is strictly decreasing in ε and it has exactly two zeros 0 and $1-\varepsilon$. $\hat{f}_\varepsilon(u)$ is strictly increasing in ε and it has exactly two zeros 0 and $1+\varepsilon$. In a similar way as above, we know that problem (2.1) with f replaced by \tilde{f}_ε (resp. \hat{f}_ε) has exactly one solution $(\tilde{c}_r^*, \tilde{q}_r^*)$ with $\mu(\tilde{q}_r^*)'(0) = \tilde{c}_r^*$ and $\tilde{c}_r^* \in (0, c_r^0)$ (resp. a solution $(\hat{c}_r^*, \hat{q}_r^*)$ with $\mu(\hat{q}_r^*)'(0) = \hat{c}_r^*$ and $\hat{c}_r^* \in (0, c_r^0)$), where $c_r^0 = 2\sqrt{d} + \beta$. Similarly, problem (2.5) with f replaced by \tilde{f}_ε (resp. \hat{f}_ε) has exactly one solution $(\tilde{c}_l^*, \tilde{q}_l^*)$ with $\mu(\tilde{q}_l^*)'(0) = \tilde{c}_l^*$ and $\tilde{c}_l^* \in (0, c_l^0)$ (resp. a solution $(\hat{c}_l^*, \hat{q}_l^*)$ with $\mu(\hat{q}_l^*)'(0) = \hat{c}_l^*$ and $\hat{c}_l^* \in (0, c_l^0)$), where $c_l^0 = 2\sqrt{d} - \beta$.

Proposition 2.5 *The following conclusions hold.*

$$\tilde{c}_r^* < c_r^* < \hat{c}_r^*, \quad \lim_{\varepsilon \rightarrow 0} \tilde{c}_r^* = \lim_{\varepsilon \rightarrow 0} \hat{c}_r^* = c_r^*,$$

and

$$\tilde{c}_l^* < c_l^* < \hat{c}_l^*, \quad \lim_{\varepsilon \rightarrow 0} \tilde{c}_l^* = \lim_{\varepsilon \rightarrow 0} \hat{c}_l^* = c_l^*.$$

Proof. We first prove $\tilde{c}_r^* < c_r^*$. For any $c \in [0, c_r^0)$, consider the problem (2.3) with f replaced by \tilde{f}_ε , denote the solution with trajectory through the critical point $(0, 1-\varepsilon)$ in the phase plane by $\tilde{P}_{r,\varepsilon}^c(q)$. Similar as Proposition 2.1 (i) we

have $\tilde{P}_{r,\varepsilon}^c(0) > 0$ for all $c \in [0, c_r^0]$. Moreover, $\tilde{P}_{r,\varepsilon}^c(q) < P_r^c(q)$ ($q \in (0, 1 - \varepsilon]$) by $\tilde{f}_\varepsilon(q) \leq f(q)$ ($0 < q \leq 1 - \varepsilon$). We now prove

$$0 < \tilde{P}_{r,\varepsilon}^c(0) < P_r^c(0) \quad \text{for } c \in [0, c_r^0]. \quad (2.8)$$

Otherwise, $\tilde{P}_{r,\varepsilon}^c(0) = P_r^c(0)$, and so the function $\eta(q) := P_r^c(q) - \tilde{P}_{r,\varepsilon}^c(q)$ satisfies

$$\eta' < a(q)\eta \quad (0 < q < 1 - \varepsilon), \quad \eta(0) = 0,$$

where $a(q) := \tilde{f}_\varepsilon(q)[dP_r^c(q)\tilde{P}_{r,\varepsilon}^c(q)]^{-1}$. This implies that $\eta(q) < 0$ ($0 < q < 1 - \varepsilon$), a contradiction.

Denote $\tilde{\zeta}(c) := \tilde{P}_{r,\varepsilon}^c(0) - \frac{c}{\mu}$. Then (2.8) implies that

$$\tilde{\zeta}(c) < \zeta(c) \quad \text{for } c \in [0, c_r^0].$$

Similar as above, both $\tilde{\zeta}(c)$ and $\zeta(c)$ are continuous and strictly decreasing functions in $[0, c_r^0]$, and

$$\tilde{\zeta}((c_r^0)^-) = \zeta((c_r^0)^-) = -\frac{c_r^0}{\mu}.$$

Therefore $\tilde{c}_r^* < c_r^*$ by their definitions: $\tilde{\zeta}(\tilde{c}_r^*) = \zeta(c_r^*) = 0$.

Next we prove $\lim_{\varepsilon \rightarrow 0} \tilde{c}_r^* = c_r^*$. It is sufficient to show that, for any $c \in [0, c_r^0]$,

$$\tilde{P}_{r,\varepsilon}^c(0) \rightarrow P_r^c(0) \quad \text{as } \varepsilon \rightarrow 0. \quad (2.9)$$

By the monotonicity of \tilde{f}_ε , it is easily seen that $\tilde{P}_{r,\varepsilon}^c(q)$ is monotonically decreasing in ε , and it is bounded from above by $P_r^c(q)$. Therefore, as $\varepsilon \rightarrow 0$, $\tilde{P}_{r,\varepsilon}^c(q)$ converges to some function $R(q)$ in $C^1([0, 1 - \delta])$ for any $0 < \delta < 1$. Clearly, $p = R(q)$ corresponds to a trajectory of (2.2) that approaches $(1, 0)$ in the phase plane with a non-positive slope at $(1, 0)$. Consequently, $R(q) \equiv P_r^c(q)$, and so (2.9) is proved.

Other conclusions can be proved in a similar way as above. \square

2.2 Asymptotic spreading speed

Proof of Theorem 1.1. We first estimate the rightward asymptotic spreading speed. For any small $\varepsilon > 0$ we define

$$\tilde{w}(t, x) := \tilde{q}_r^*(\tilde{c}_r^*t - x), \quad x \in [0, \tilde{c}_r^*t],$$

Since $(\tilde{c}^*, \tilde{q}_r^*(z))$ satisfies

$$\begin{cases} q'' - \frac{c-\beta}{d}q' + \frac{\tilde{f}_\varepsilon(q)}{d} = 0 & \text{for } z \in (0, \infty), \\ q(0) = 0, \quad q(\infty) = 1 - \varepsilon, \quad q'(0) = \frac{c}{\mu} \text{ and } q'(z) > 0 \quad (z > 0), \end{cases}$$

we have

$$\tilde{w}(t, x) \leq 1 - \varepsilon, \quad \tilde{w}_t - \tilde{w}_{xx} + \beta \tilde{w}_x = \tilde{f}_\varepsilon(\tilde{w}) \quad \text{for } x \in [0, \tilde{c}_r^* t], \quad t > 0,$$

and

$$\tilde{w}(t, \tilde{c}_r^* t) = 0, \quad \tilde{c}_r^* = -\mu \tilde{w}_x(t, \tilde{c}_r^* t) \quad \text{for } t \geq 0.$$

Since we are considering the spreading case, we have $\lim_{t \rightarrow \infty} u(t, \cdot) = 1$ locally uniformly in \mathbb{R} . In particular,

$$u(t, 0) > 1 - \varepsilon \quad \text{for } t > T$$

for some $T > 0$. Thus $(\tilde{w}(t, x), \tilde{c}_r^* t)$ is a lower solution of (P_1) on $\{(t, x) \mid x \in [0, \tilde{c}_r^* t], \quad t > 0\}$ by comparison principle (cf. [6, 7]), and

$$\tilde{c}_r^* t \leq h(t + T), \quad \tilde{w}(t, x) \leq u(t + T, x) \quad \text{in } \{(t, x) \mid x \in [0, \tilde{c}_r^* t], \quad t > 0\}.$$

This implies that

$$\liminf_{t \rightarrow \infty} \frac{h(t)}{t} \geq \tilde{c}_r^*. \quad (2.10)$$

Next we estimate the upper bound of the rightward spreading speed. Consider the problem

$$\eta'(t) = f(\eta) \quad (t > 0), \quad \eta(0) = \|u_0\|_\infty + 1.$$

A simple comparison shows that

$$u(t, x) \leq \eta(t) := \left(1 - \frac{\|u_0\|_\infty}{\|u_0\|_\infty + 1} e^{-t}\right)^{-1} \quad \text{for } x \in [g(t), h(t)], \quad t > 0.$$

Hence for any small $\varepsilon > 0$, there exists $\hat{T} > 0$ such that

$$u(t, x) \leq 1 + \frac{\varepsilon}{2} \quad \text{for } x \in [0, h(t)], \quad t \geq \hat{T}.$$

Recall that $(\hat{c}_r^*, \hat{q}_r^*(z))$ is a solution of problem (2.1) with f replaced by \hat{f}_ε and $\hat{q}_r^*(\infty) = 1 + \varepsilon$. Hence there exists $\hat{x} > h(\hat{T})$ large such that

$$u(\hat{T}, x) \leq 1 + \frac{\varepsilon}{2} < \hat{q}_r^*(\hat{x} - x) \quad \text{for } x \in [0, h(\hat{T})].$$

Define

$$\widehat{w}(t, x) := \widehat{q}_r^*(\widehat{c}_r^* t + \widehat{x} - x) \quad \text{for } x \in [0, \widehat{c}_r^* t + \widehat{x}], \quad t > 0.$$

Then $(\widehat{w}, \widehat{c}_r^* t + \widehat{x})$ is an upper solution of (P_1) on $\{(t, x) \mid x \in [0, h(t + \widehat{T})], \quad t > 0\}$, and by the comparison principle (cf. [6, 7]) we have

$$h(t + \widehat{T}) \leq \widehat{c}_r^* t + \widehat{x}, \quad u(t + \widehat{T}, x) \leq \widehat{w}(t, x) \quad \text{for } x \in [0, h(t + \widehat{T})] \text{ and } t > 0.$$

This implies that

$$\limsup_{t \rightarrow \infty} \frac{h(t)}{t} \leq \widehat{c}_r^*. \quad (2.11)$$

Since the limits (2.10) and (2.11) hold for any small $\varepsilon > 0$, we have

$$\lim_{t \rightarrow \infty} \frac{h(t)}{t} = c_r^*$$

by Proposition 2.5. The leftward spreading speed

$$\lim_{t \rightarrow \infty} \frac{-g(t)}{t} = c_l^*$$

is proved similarly.

In [6, 7], the authors considered problem (P_0) , that is, problem (P_1) *without advection term* (i.e. $\beta = 0$). Among others, they showed that the asymptotic spreading speed is characterized by the following problem

$$\frac{dp}{dq} = \frac{c}{d} - \frac{f(q)}{dp} \quad (q < 1), \quad p(1^-) = 0. \quad (2.12)$$

Using a similar phase plane analysis as above, the authors in [7] proved that problem (2.12) has a solution $(c, P^c(q))$ for each $c \in [0, 2\sqrt{d}]$. Moreover, they proved that $P^c(0) \searrow 0$ as $c \nearrow c^0 := 2\sqrt{d}$; (2.12) has a unique solution $(c^*, P^{c^*}(q))$ such that $\mu P^{c^*}(0) = c^*$. This c^* is nothing but the rightward and leftward spreading speeds (cf. [6, 7]).

Combining the above phase plane analysis we have the following conclusions:

1. $P_l^{c-\beta}(0) = P^c(0) = P_r^{c+\beta}(0)$ for all $c \in [\beta, 2\sqrt{d}]$;
2. $P_l^c(0)$ (resp. $P^c(0), P_r^c(0)$) is continuous and strictly decreasing in $c \in [0, 2\sqrt{d} - \beta]$ (resp. $c \in [0, 2\sqrt{d}]$, $c \in [0, 2\sqrt{d} + \beta]$).

Define three new functions $\gamma_r(c)$, $\gamma(c)$ and $\gamma_l(c)$ by

$$\gamma_r(c) := P_r^c(0) \text{ for } c \in [0, 2\sqrt{d} + \beta), \quad \gamma(c) := P^c(0) \text{ for } c \in [0, 2\sqrt{d})$$

and

$$\gamma_l(c) := P_l^c(0) \text{ for } c \in [0, 2\sqrt{d} - \beta).$$

Then, in the c, γ -plane their graphes lie in the first quadrant (see Figure 1) and these graphes contact the straight line $\gamma = \frac{c}{\mu}$ at points $(c_l^*, \frac{c_l^*}{\mu})$, $(c^*, \frac{c^*}{\mu})$ and $(c_r^*, \frac{c_r^*}{\mu})$, respectively. Therefore, $c_l^* < c^* < c_r^*$. This completes the proof of Theorem 1.1. \square

Proof of Theorem 1.2. The conclusions in Theorem 1.2 follow from a simple analysis on the relations among the graphes of $\gamma_r(c)$, $\gamma(c)$, $\gamma_l(c)$ and c/μ in Figure 1. \square

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